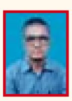
Multiple Critical Paths and Project Crashing: Identification of Combinations Using Theory of Partition of Numbers

The original developers of the project management technique called Critical Path Method (CPM) provided the project manager with the option of adding resources to selected activities to reduce project completion time. Added resources (such as more workers, overtime, and so on) generally increase project costs, so the decision to reduce activity times must take into consideration the additional cost involved. In effect, the project manager has to make a decision that involves trading reduced activity for additional project cost. The shortening of activity times, which usually can be achieved by adding resources, is referred to as crashing.

The critical path method is used in arriving at a solution for crashing the cost of a project. The project duration is controlled by the critical path. Hence we reduce that activity on the critical path having the least cost slope by the maximum crash duration possible. The reduction is carried out in such a manner that the critical path remains as critical path. In this process one or more paths may become critical. When there are more than one critical path or if all the paths are critical a combination of one activity from each path is necessary for the purpose of reduction subject to the cost slope being minimum. To understand the Combinations let us consider the following situations of network.

Situation 1: Network consisting of only independent activities.

Situation 2: Network consisting of independent activities and a common activity.




— Dr. P. R. Vittal*
— CA. R. Sivakumar**

*(*The author is Retd. Principal, Ramakrishna Mission Vivekananda College, Chennai. **The author is a Member of the Institute. They can be reached at sivakumar.karra@gmail.com)*

Situation 3: Network consisting of independent activities and activities common to few of the paths.

Situation 4: Network consisting of independent activities, activities common to all the paths and activities common to few of the paths.

Characteristics of Combination

The following characteristics hold good as regards combination:

1. If there are independent activities in each path, a combination or combinations will emerge among these activities.
2. If an activity is common to all the paths that activity itself will be a combination.
3. If an activity is not common to all the paths but only for some of the paths it can be combined with the independent activity of the other paths or it can also be combined with the common activity of the other paths.

NOTE: The maximum number of activities forming a combination will always be less than or equal to the total number of paths.

Theory of Partition of Numbers

A basic problem of a specific branch of number theory called additive number theory is that of expressing a given positive integer n as a sum of integers from some given set A, say

$A = \{a1, a2 \ldots\}$

where the elements ai are special numbers such as primes, squares, cubes, triangular numbers etc. Each representation of n as a sum of elements of A is called a partition of n and we are interested in the arithmetical function $A(n)$ which counts the number of partitions of n into summands taken from *A*.

One of the most fundamental problems in additive number theory is that of unrestricted partitions. The set of summands consists of all positive integers, and the partition function to be studied is the number of ways *n* can be written as a sum of positive integers $\leq n$, that is, the number of solutions of

$N = ai1 + ai2 + \ldots$

The number of summands is unrestricted, repetition is allowed, and the order of the summands is not taken into account. The corresponding partition function is denoted by *p(n)* and is called the unrestricted partition function, or simply the partition function. The summands are called parts. For example, there are exactly five partitions of 4, given by

$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$

With respect to the combination of activities for the purpose of crashing it can be noticed that since the number of activities to be considered equals the total number of paths the selection of activities constitutes a partition. For example if there are four paths the combination is as follows:

1. One activity from each path (1 + 1 + 1 + 1 = 4)
2. An activity, which is common to two paths and two independent activities (2+1+1= 4)
3. Two activities which are common to two different paths (2 + 2 = 4)
4. Activity common to three of the paths and an independent activity (3 + 1 = 4)
5. Activity which is common to all the paths (4)

Allocation of numbers and analysis of the partition of the same for proving the existence of a Combination

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If one wants to use the idea of partitions of a number the best would be to use a partition of the number that represents the total number of paths in the network. It may be noted that though the nodes may vary the number of paths may still be the same. Let us now assume that there are *n* paths in the network. It is therefore natural to use the partition of the number *n* or at least partitions of a multiple of *n*. Instead of taking an arbitrary multiple of the number *"n"* we go for a partition of n^2. Instead of considering an arbitrary partition of the number n^2 we shall take a particular one. The partition is 1,3,5,7,…,(2n-1).

To show that this is a partition needs only a simple computation. Now

$1+3+5+\ldots+(2n-1) = (1+2+3+\ldots+2n)-(2+4+6+\ldots+2n) = (1+2+3+\ldots+2n)-2(1+2+3+\ldots+n) = (2n(2n+1)/2))-2(n(n+1)/2) = 2n^2+n-n^2-n = n^2$.

Based on the above analysis the following steps emerge:

Step 1: We assign the number in the order of 1,3,5 etc. for each of the paths. If there are '*n*' paths the assigning will be 1,3,5,7…,…,…, (2n-1). If there are 3 paths we assign the numbers 1,3 and 5 respectively.

Step 2: On the basis of the numbers assigned to the paths in step no.1 we compute the weight for each of the activity. The weight of an independent activity will equal the number of the paths in which it is independent. The weight of a common activity of all the paths will equal the sum of all the numbers of the paths. The weight of an activity, which is common to few of the paths, will equal the sum of the numbers of the paths of which it is common.

Step 3: It can be established that where the total relationship of the activities of the paths equal the total number of the paths (3) and the corresponding total of the weights of the activities equal the square of the total number of the paths (9), a combination emerges.

Now let us analyse any combination. If all the activities of a combination are independent then clearly the weights of the activities (the weight of the path itself) is precisely the sum of all the weights, which is therefore n^2. On the other hand if an activity is common to more than one path then while taking that activity into account for a combination we are actually taking into account the same activity (being common) from each of the paths for which it is common. In this case the weights of the activities forming the combination is once again the sum of all the weights (the common activity giving the weights of the paths for which it is common and the other individual activities giving the other respective weights). So once again the total weight is n^2. Thus it may be noted that if we form a combination of activities for the said purpose of reduction the total weight must be n^2.

This general idea can be used with any weights of the paths forming a partition of a particular number. i.e, using any partition of a number which yields a set of *n* numbers that could be distributed to the *n* paths in the network we can use the idea that a combination should be such that the total weights should add to the number for which the *n* numbers form a partition. In this context it may be noted that 1,3,5,…, (2n-1) clearly forms a partition of n^2 and there are *n* numbers in the partition.

Conclusion

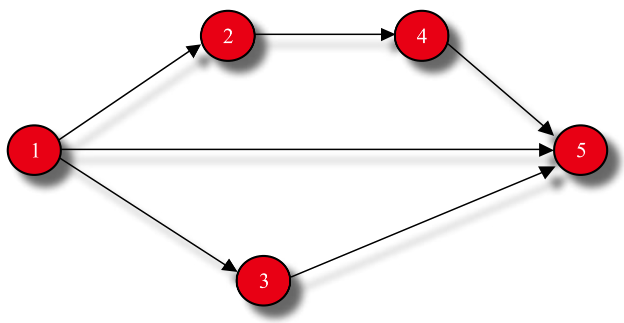
Many researchers have done lotofworkandnoteworthyamong them is Srinivasa Ramanujan, the Indian mathematical genius who has given an excellent formula for calculating the partitions of a given number. It should be noted that any network can be represented as a square matrix, and any square can be partitioned in the natural ordering. Hence partitions become the natural choice for our Critical Path Method.

This article is confined only to a part of the subject of Project Crashing viz., various combinations that emerge when all the paths are critical. The cost and time are not considered. By applying the theory of partition the various combinations are established. However, a complex project situation may throw open different dimensions of the combinations itself.

Illustrative Examples of Combinations

Situation 1: Network with various paths having only independent activities: In this case the number of independent activities in each of the paths is computed and multiplied to arrive at the total number of combinations.

Example:



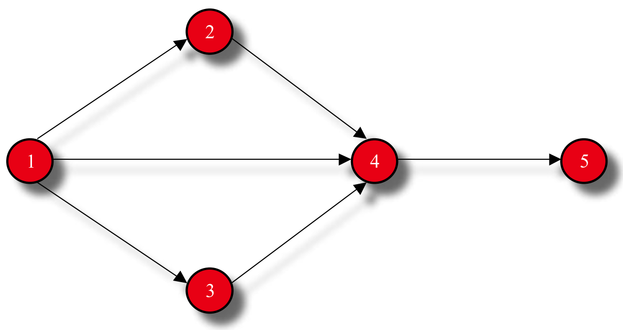
		Weight of the path	ACTIVITY						Total Activity on the path
			12	13	15	24	35	45	
No. of occurrences on the path			1	1	1	1	1	1	
Paths	1245	1	I (1)			I (1)		I (1)	I(3)
	135	3		I (3)			I (3)		I(2)
	15	5			I (5)				I (1)
Total weight of activities			1	3	5	1	3	1	I (6)
*Overall no. of combinations 3*2*1 = 6*									

(Where I= Independent activity. The numbers in brackets under activity column represents weights, and in total activity on the path column represents total number of activities in the path.)

COMBINATION OF ACTIVITIES	12 13 15	12 35 15	24 13 15	24 35 15	45 13 15	45 35 15
TOTAL OCCURRENCES	I(3) = 3	I(3) = 3	I(3) = 3	I(3) = 3	I(3) = 3	I(3) = 3
TOTAL WEIGHT	1+3+5 =9	1+3+5 =9	1+3+5 =9	1+3+5 =9	1+3+5 =9	1+3+5 =9

NOTE: The total weight is also a partition of the number 9.

Situation 2: Network consisting of paths with independent activities and also activities common to all the paths:



In this case, the number of combinations will be as computed in Situation 1 plus the number of common activities.

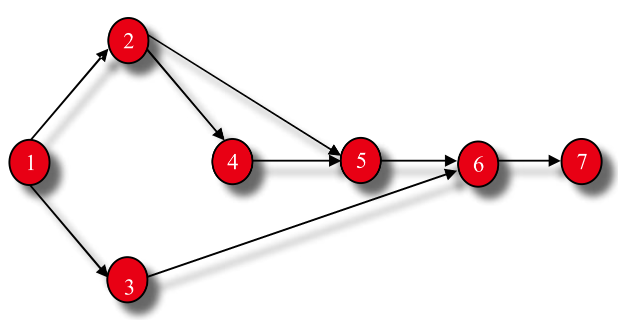
		Weight Of The Path	ACTIVITY						Total Activity On The Path
			12	13	14	24	34	45	
No. Of Occurrences On The Path			1	1	1	1	1	3	
Paths	1-2-4-5	1	I (1)			I (1)		C (1)	I(2) C(1)
	1-3-4-5	3		I (3)			I (3)	C (3)	I(2) C(1)
	1-4-5	5			I (5)			C (5)	I(1) C(1)
Total			1	3	5	1	3	9	I(5) C(1)
*Overall no. of combinations (2*2*1 = 4)+1 = 5*									

(I = independent activity and C = Common activity. The numbers in brackets under activity column represents weights, and in total activity on the path column represents the total number of activities in the path.)

COMBINATION OF ACTIVITIES	12 13 14	12 34 14	24 13 14	24 34 14	45
TOTAL OCCURRENCES	I(3) = 3	I(3) = 3	I(3) = 3	I(3) = 3	C(3) = 3
TOTAL WEIGHT	1+3+5 =9	1+3+5 =9	1+3+5 =9	1+3+5 =9	9

Situations 3 & 4: Network consisting of paths with independent activities, activities common to some of the paths only and common to all the paths.

Unlike situations 1 and 2, where it is easy to identify the number of combinations a difficulty arises in situations 3 and 4 since the combinations can be arrived at only by heuristic methods. By inspection the number of combinations equal (2*1*2)=4 + ((2*1+2*1)=4+1=9



		Weight of the path	ACTIVITY								Total Activity on the path		
			12	13	24	25	36	45	56	67			
No. of occurrences on the path			2	1	1	1	1	1	2	3	I	C (All)	C (Few)
Activity	124567	1	C (1)		I (1)			I (1)	C (1)	C (1)	2	1	2
	12567	3	C (3)			I (3)			C (3)	C (3)	1	1	2
	1367	5		I (5)			I (5)			C (5)	2	1	
			4	5	1	3	5	1	4	9	5	1	2
Total number of combinations (2*1*2)=4 + ((2*1+2*1)=4+1=9													
(I = independent activity and C = Common activity. The numbers in brackets under activity column represents weights.)													

COMBINATION OF ACTIVITIES	67	12 13	12 36	24 25 13	24 25 36	56 36	56 13	45 25 36	45 25 13
TOTAL OCCURRENCES	C (3)	C(2) +I(1) =3	C(2) +I(1) =3	I (3) =3	I (3) =3	C(2) +I(1) =3	C(2) +I(1) =3	I (3) =3	I (3) =3
TOTAL WEIGHT	9	4+5 =9	4+5 =9	1+3+5 =9	1+3+5 =9	4+5 =9	4+5 =9	1+3+5 =9	1+3+5 =9

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The Chairman,
Toronto Chapter of the Institute of
Chartered Accountants of India,
4252 Bloor Street West, Toronto, ON, M9C 1Z7, Canada
Tel: +1(416) 601 5934
Email: hishah@deloitte.ca

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Members in Toronto, Canada desirous of becoming members of the Chapter may contact the following:

CA. Himanshu Shah, Chairman - Email: hishah@deloitte.ca

CA. Ram Balakrishnan, Secretary - Email: rbalakrishnan@deloitte.ca